ON THE SINGULARITIES OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

BY

LARS HÖRMANDER

ABSTRACT

Let P be a differential operator with constant coefficients in $\mathbb{R}^n$. If u is a distribution, the singular support of u is the complement of the largest set where $u \in C^\infty$. Necessary and sufficient conditinos are obtained for a closed convex set Γ to be equal to the singular support of u for some u with $Pu \in C^\infty$ or, equivalently, for Γ to contain the singular support of u for some u with $Pu \in C^\infty$ but $u \notin C^\infty$. Related local uniqueness theorems analogous to the Holmgren theorem with supports replaced by singular supports are also given, as well as applications concerning P-convexity with respect to singular supports.

1. Introduction

Let P be a partial differential operator with constant coefficients in $\mathbb{R}^n$. We shall here continue the study begun in [2] and [3] of the singularities of distributions u with $Pu \in C^\infty$. In particular we shall give a necessary and sufficient condition on a closed convex set $\Gamma \subset \mathbb{R}^n$ for the existence of a solution of the equation $Pu = 0$ with $\operatorname{sing\,supp} u = \Gamma$. (Here sing supp u is the smallest closed set such that $u \in C^\infty$ in the complement.) The condition only depends on the largest linear subspace V with $\Gamma + V = \Gamma$ and on the strength of P as defined in [1]. For open convex sets $X_1 \subset X_2 \subset \mathbb{R}^n$, our results also show that

$$u \in \mathscr{D}'(X_2),\ Pu \in C^\infty(X_2),\ u \in C^\infty(X_1) \Rightarrow u \in C^\infty(X) \tag{1.1}$$

if and only if all such hyperplanes intersecting X also meet X_1. This is analogous to the fact that

$$u \in \mathscr{D}'(X_2),\ Pu = 0 \text{ in } X_2,\ u = 0 \text{ in } X_1 \Rightarrow u = 0 \text{ in } X \tag{1.2}$$

if and only if all characteristic hyperplanes intersecting X also meet X_1 (see

[1, Theorem 5.3.3]). Just as (1.2) is studied by means of Holmgren's uniqueness theorem and a construction of null solutions, we examine (1.1) by means of a uniqueness theorem and a construction of solutions singular on a hyperplane.

To be more explicit we write $P = P(D)$ where P is a polynomial in n variables with complex coefficients and $D = (-i\partial/\partial x_1, \cdots, -i\partial/\partial x_n)$. Let $|\ |$ be any norm in $\mathbb{R}^n$ and set for a linear subspace V of $\mathbb{R}^n$

$$\tilde{P}_V(\xi, t) = \sup\{|P(\xi + \theta)|\ ;\ \theta \in V,\ |\theta| \leqq t\}.$$

It is obvious that

$$\tilde{P}_V(\xi, st) \leqq Cs^m \tilde{P}_V(\xi, t), \qquad s \geqq 1,$$

the constant C depending only on the degree m of P. Thus we change $\tilde{P}_V$ by at most a fixed factor when passing to another norm in $\mathbb{R}^n$. When $V = \mathbb{R}^n$ we write $\tilde{P}(\xi, t)$ instead of $\tilde{P}_V(\xi, t)$. With constants depending only on n and m we have

$$C_1\tilde{P}(\xi, t) \leqq (\Sigma |P^{(\alpha)}(\xi)|^2 t^{2|\alpha|})^{1/2} \leqq C_2\tilde{P}(\xi, t), \tag{1.3}$$

as follows from the finite dimensionality of the space of polynomials in $\mathbb{R}^n$ of degree m. (Cf. [1, sections 3.1 and 3.3].) Hence our notation agrees essentially with that used in [1] and we can change the definition to the middle term in (1.3) whenever this is more convenient.

Now set

$$\sigma_P(V) = \inf_{t>1} \liminf_{\xi \to \infty} \tilde{P}_V(\xi, t)/\tilde{P}(\xi, t). \tag{1.4}$$

This is a continuous function of V in the sense that

$$\sigma_P(V) \leqq \sigma_P(W) + Cd(V, W) \tag{1.5}$$

if

$$d(V, W) = \sup_{x \in V, |x| = 1} \inf_{y \in W, |y| = 1} |x - y|. \tag{1.6}$$

To prove (1.5) we first observe that

$$|P(\xi + \theta) - P(\xi + \eta)| \leqq C\tilde{P}(\xi, t)|\theta - \eta|/t \tag{1.7}$$

if $|\theta| \leqq t$, $|\eta| \leqq t$. If $\theta \in V$ and $|\theta| \leqq t$ we choose $\eta \in W$ with $|\eta| = |\theta|$ and $|\theta - \eta| \leqq td(V, W)$, and obtain

$$|P(\xi + \theta)| \leqq |P(\xi + \eta)| + Cd(V, W)\tilde{P}(\xi, t),$$

which gives (1.5).

THEOREM 1.1. *Let V be a linear subspace of $\mathbb{R}^n$. If there is a distribution u*

with $P(D)u \in C^\infty(\mathbb{R}^n)$ *and* $\emptyset \neq \operatorname{sing\,supp} u \subset V$ *then* $\sigma_P(V') = 0$ *where* V' *is the orthogonal space of* V. *Conversely, if* $\sigma_P(V') = 0$, *one can for every non-negative integer* μ *find* $u \in C^\mu(\mathbb{R}^n)$ *with* $P(D)u = 0$, $\operatorname{sing\,supp} u = V$ *and* $u \notin C^{\mu+1}(N)$ *if* N *is any open set intersecting* V.

Note that (1.5) implies that the set of subspaces V with $\sigma_P(V') = 0$ and of fixed dimension is closed in the standard topology of the Grassmannian, and it is independent of the choice of norm in $\mathbb{R}^n$. Theorem 1.1 and additional uniqueness theorems will be proved in sections 2, 3 and 4.

In connection with a study of results of the form (1.1), F. John [6] has discussed "Hölder estimates" for solutions of certain partial differential equations, mainly elliptic ones. These are estimates of some semi-norms of u in X by geometric means of semi-norms of u in X_1 and in X_2 which are valid for the solutions of the equation $P(D)u = 0$ in X_2. In section 5 we add a study of such estimates. They are closely related to the properties of P as an operator in $\mathbb{R}^{n+k}$ when the last k variables are regarded as parameters. Various properties of σ_P and some examples are discussed in section 6.

2. Preliminaries

The definition (1.4) of $\sigma_P(V)$ is not suitable for the proofs. In fact, t and $\log|\xi|$ will always have comparable size there. To obtain an equivalent but more useful definition we shall apply the Tarski-Seidenberg theorem (see [1, appendix]).

LEMMA 2.1. *If* $\sigma_P(V) = 0$, *it follows that there are positive constants* b, β, r_1, ρ *such that for any* $t > 1$ *and* $r > r_1 t^\rho$ *one can find* $\xi \in \mathbb{R}^n$ *with* $|\xi| = r$ *and*

$$\tilde{P}_V(\xi, t) < b t^{-\beta} \tilde{P}(\xi, t). \tag{2.1}$$

LEMMA 2.2. *If* $\sigma_P(V) \neq 0$, *it follows that there are positive constants* b, r_1, ρ *such that for* $t > 1$ *and* $|\xi| > r_1 t^\rho$ *we have*

$$\tilde{P}_V(\xi, t) > b \tilde{P}(\xi, t). \tag{2.2}$$

PROOF OF LEMMA 2.1. A routine application of the Tarski-Seidenberg theorem shows that the continuous function

$$a(t) = \liminf_{\xi \to \infty} \tilde{P}_V(\xi, t)/\tilde{P}(\xi, t)$$

is an algebraic function of t for large t. Since $a(st) \leqq C s^m a(t)$ for $s \geqq 1$ and

$\inf_{t>1} a(t) = \sigma_P(V) = 0$, it follows that $\liminf_{t\to\infty} a(t) = 0$. Hence the Puiseux series expansion of $a(t)$ shows that

$$a(t)t^{\beta} < b, \qquad t > 1$$

for some rational number β and constant $b > 0$. The set

$$M = \{(r,t);\, t > 1, \tilde{P}_V(\xi,t) < bt^{-\beta}\tilde{P}(\xi,t) \text{ for some } \xi \in \mathbb{R}^n, |\xi| = r\}$$

is semi-algebraic by the Tarski-Seidenberg theorem, and if $t > 1$ we have $(r,t) \in M$ for arbitrarily large values of r. It follows that there is a piecewise algebraic function $r(t)$ such that $(r,t) \in M$ if $r > r(t)$ and $t > 1$, which proves the lemma.

Proof of Lemma 2.2. Let $0 < b < \sigma_P(V)$. Then the function

$$a(t) = \sup\{|\xi|;\, \tilde{P}_V(\xi,t) < b\tilde{P}(\xi,t)\}$$

is finite for $t > 1$ and it is a piecewise algebraic function of t by the Tarski-Seidenberg theorem. Hence there are positive constants ρ, r_1 such that $a(t) < r_1 t^{\rho}$ if $t > 1$, which proves Lemma 2.2.

By combining the lemmas we conclude that $\sigma_P(V) = 0$ if and only if

$$\liminf_{\xi\to\infty} \tilde{P}_V(\xi, \lambda\log|\xi|)/\tilde{P}(\xi,\lambda\log|\xi|) = 0 \tag{2.3}$$

for some (for all) $\lambda > 0$. This is the form required in what follows. Note that (2.3) is related to the localizations at infinity used for example in [5].

3. Solutions with a convex singular support

In this section we shall prove a result containing the second half of Theorem 1.1:

Theorem 3.1. *Let Γ be a closed convex set in $\mathbb{R}^n$ and V a linear subspace of $\mathbb{R}^n$ with $\Gamma + V = \Gamma$. If $\sigma_P(V') = 0$, one can for every non-negative integer μ find $u \in C^{\mu}(\mathbb{R}^n)$ with $P(D)u = 0$, $\operatorname{sing\,supp} u = \Gamma$ and $u \notin C^{\mu+1}(N)$ for every open set N intersecting Γ.*

For the proof, we denote by F the set of all $u \in C^{\mu}(\mathbb{R}^n)$ with $P(D)u = 0$ such that $u \in C^{\infty}(\mathbb{R}^n \setminus \Gamma)$. With the weakest topology making the inclusion $F \to C^{\mu}(\mathbb{R}^n)$ and the restriction $F \to C^{\infty}(\mathbb{R}^n \setminus \Gamma)$ continuous, F is a Fréchet space. To prove the theorem, it suffices to show that $\{u \in F;\, u \in C^{\mu+1}(N)\}$ is of the first category in F for every open set N with $N \cap \Gamma \neq \emptyset$, for we need only consider countably many such sets N. It is no restriction to assume that $0 \in N \cap \Gamma$. If Theorem 3.1 were false, it would therefore follow from the closed graph theorem that for some such N there is a continuous restriction map $F \to C^{\mu+1}(N)$. Thus there must exist compact sets $K_1 \subset \mathbb{R}^n$ and $K_2 \subset \mathbb{R}^n \setminus \Gamma$ and an integer ν such that

$$\sum_{|\alpha|=\mu+1} |D^\alpha u(0)| \leqq C\Big(\sum_{|\alpha|\leqq\mu} \sup_{K_1} |D^\alpha u| + \sum_{|\alpha|\leqq\nu} \sup_{K_2} |D^\alpha u| \Big), \qquad u \in F. \tag{3.1}$$

Theorem 3.1 will therefore be proved if we show that this estimate cannot hold if $\sigma_P(V') = 0$ and $K_2 \subset \mathbb{R}^n \setminus V$. To do so we shall take u as a superposition of exponential solutions which is estimated by means of the following two lemmas contained in Lemmas 2.2 and 2.3 of [4].

LEMMA 3.2. *There exists a sequence of functions $\phi^N \in C_0^\infty(\mathbb{R})$ such that*

(i) $\operatorname{supp}\phi^N \subset (-1,1)$; $\phi^N \geqq 0$, $\int \phi^N dt = 1$;

(ii) $\int |d^k\phi^N/dt^k| \, dt \leqq (2N)^k$, $0 \leqq k \leqq N$; $N = 1,2,\cdots$.

In the following lemma we write

$$\Phi_R^N(\xi) = R^{-\nu}\phi^N(\xi_1/R)\cdots\phi^N(\xi_\nu/R), \ \xi \in \mathbb{R}^\nu, \ R > 0,$$

and we set $|\xi| = \max|\xi_j|$.

LEMMA 3.3. *Let F be an analytic function with $|F| \leqq M$ in the polydisc $\Omega_R = \{\zeta \in \mathbb{C}^\nu, |\zeta| < 2R\}$, and set*

$$u^N(x) = \int e^{i\langle x,\xi\rangle} F(\xi)\Phi_R^N(\xi)d\xi.$$

Then $u^N(0) = 1$ if $F = 1$ and in general we have

$$|x|^k |u^N(x)| \leqq (3N/R)^k M, \qquad 0 \leqq k \leqq N. \tag{3.2}$$

Next we shall discuss how to produce suitable exponential solutions to average when $\tilde{P}_{V'}(\xi,t)/\tilde{P}(\xi,t)$ is small. We can change the coordinate system so that V is defined by $x' = 0$ if $x = (x',x'')$, $x' = (x_1,\cdots,x_\nu)$, $x'' = (x_{\nu+1},\cdots,x_n)$ is a splitting of the coordinates in two groups. By (1.3) we can change the definition of $\tilde{P}$ and $\tilde{P}_{V'}$ so that from now on we take instead

$$\tilde{P}(\xi,t) = (\Sigma|P^{(\alpha)}(\xi)|^2 t^{2|\alpha|})^{1/2}, \quad \tilde{P}'(\xi,t) = (\sum_{\alpha''=0}|P^{(\alpha)}(\xi)|^2 t^{2|\alpha|})^{1/2}.$$

LEMMA 3.4. *For suitable positive constants ε_0, C, γ (depending only on n and m) the inequality*

$$\tilde{P}'(\xi,t)/\tilde{P}(\xi,t) \leqq \varepsilon < \varepsilon_0 \tag{3.3}$$

implies that there exists an analytic map $\theta \to \zeta(\theta)$ from $\Omega_{\gamma t}$ to $\mathbb{C}^n$ such that

(i) $\zeta'(\theta) = \xi_0' + \theta$ *where* $\xi_0' \in \mathbb{R}^\nu$ *and* $|\xi_0' - \xi'| \leqq t$,

(ii) $|\zeta''(\theta) - \xi''| \leqq Ct\varepsilon^{1/m}$, $\theta \in \Omega_{\gamma t}$,

(iii) $P(\zeta(\theta)) = 0$.

PROOF. It is no restriction to assume that $t = 1$ and that $\xi = 0$. If $n > \nu + 1$ we choose a finite number of vectors $\theta_1, \cdots, \theta_k \in \mathbb{R}^{n-\nu}$ such that no polynomial of degree m vanishes in $\mathbb{R}^\nu \times (\mathbb{R}\theta_j) \subset \mathbb{R}^n$ for every j without vanishing identically. Some of the polynomials

$$\mathbb{R}^{\nu+1} \ni \xi \to P(\xi', \xi_{\nu+1}\theta_j)$$

will then satisfy the hypotheses of the lemma with ε replaced by a constant times ε. We may therefore assume that $n = \nu + 1$.

By (3.3) with $\xi = 0$, $t = 1$, we have for a suitable normalization of P

$$C_1 \leqq \sup\{|P(\xi', \zeta_n)|;\ |\xi'| \leqq 1,\ |\zeta_n| \leqq 1\};\ |P(\xi', 0)| \leqq C_2\varepsilon, |\xi'| \leqq 1.$$

(We use the notation ξ for real variables and ζ for complex variables.) By the maximum principle it follows that for $0 < \delta < 1$

$$C_1\delta^m \leqq \sup\{|P(\xi', \zeta_n)|;\ |\xi'| \leqq 1,\ |\zeta_n| \leqq \delta\} = M_\delta.$$

We have

$$\sum_1^{n-1} |\partial P(\zeta)/\partial\zeta_j| \leqq C_3 M_\delta;\ |\zeta'| \leqq 2,\ |\zeta_n| \leqq \delta,$$

and in view of Lemma 3.1.7 in [1], there is some real η' with $|\eta'| \leqq 1$ and r with $0 < r < \delta$ such that

$$|P(\eta', \zeta_n)| \geqq C_4 M_\delta \text{ when } |\zeta_n| = r.$$

It follows that

(3.4) $|P(\zeta', \zeta_n)| \geqq C_4 M_\delta/2 \geqq C_4 C_1 \delta^m/2$ if $|\zeta_n| = r$ and $|\zeta' - \eta'| C_3 < C_4/2$.

If we choose δ so that $C_4 C_1 \delta^m/2 = 2C_2\varepsilon$, the equation $P(\xi', \zeta_n) = 0$ must have at least one root ζ_n with $|\zeta_n| < r \leqq \delta$ for every real ξ' with $|\xi' - \eta'| \leqq C_4/2C_3$. By Lemma A.2 in [4], the polydisc $\{\zeta';\ |\zeta' - \eta'| < C_4/2C_3\}$ contains a polydisc with real center ξ_0', $|\xi_0'| \leqq 1$, and fixed radius 2γ where the discriminants with respect to ζ_n of the irreducible factors of P are all different from 0. In view of (3.4), it follows that for ζ' in this polydisc, we can choose an analytic function $\zeta_n(\zeta')$ with $P(\zeta', \zeta_n) = 0$ and $|\zeta_n| \leqq r \leqq \delta$. The lemma is proved.

PROOF OF THEOREM 3.1. Recall that we only have to show that (3.1) does not hold for any v and compact sets $K_1 \subset \mathbb{R}^n$ and $K_2 \subset \mathbb{R}^n \backslash V$ provided that (2.3) is valid with V replaced by V'. Let λ be a fixed large positive number and let $\varepsilon > 0$. We can then find ξ and t arbitrarily large so that

$$t = \lambda \log |\xi|, \ \tilde{P}'(\xi, t)/\tilde{P}(\xi, t) < \varepsilon.$$

According to Lemma 3.4, we take an analytic solution $\zeta(\theta)$ of the equation $P(\zeta(\theta)) = 0$, $\theta \in \Omega_{\gamma t}$, and set with $R = \gamma t$ and an integer N to be chosen later

$$u(x) = \int e^{i\langle x, \zeta(\theta)\rangle} \Phi_R^N(\theta) d\theta. \tag{3.5}$$

It is clear that $P(D)u = 0$, and

$$D^\alpha u(0) = \int \zeta(\theta)^\alpha \Phi_R^N(\theta) d\theta = \xi^\alpha + O(t|\xi|^{|\alpha|-1})$$

so that

$$|\xi|^{\mu+1} \leqq C \sum_{|\alpha| \leqq \mu+1} |D^\alpha u(0)|.$$

Using (ii) in Lemma 3.4 we obtain

$$\sum_{|\alpha| \leqq \mu} \sup_{K_1} |D^\alpha u| \leqq C \exp(Ct\varepsilon^{1/m})(1 + |\xi|)^\mu \leqq C_1(1 + |\xi|)^{\mu+1/2}$$

if $C\lambda\varepsilon^{1/m} < 1/2$.

It remains to consider the last sum in (3.1). To do so, we estimate

$$D^\alpha u(x) = e^{i\langle x', \xi_0\rangle + i\langle x'', \xi''\rangle} \int e^{i\langle x', \theta\rangle} \zeta(\theta)^\alpha e^{i\langle x'', \zeta''(\theta) - \xi''\rangle} \Phi_R^N(\theta) d\theta$$

using Lemma 3.3. Since $|x'| > \delta > 0$ in K_2 this gives

$$S = \sum_{|\alpha| \leqq v} \sup_{K_2} |D^\alpha u(x)| \leqq C(3N/\delta\gamma t)^N |\xi|^v \exp(Ct\varepsilon^{1/m}). \tag{3.6}$$

We choose $N = [\delta\gamma t/3e]$ and obtain

$$S \leqq C \exp t(v/\lambda + C\varepsilon^{1/m} - \delta\gamma/3e)$$

which is bounded as $\xi \to \infty$ if $\lambda > 6ev/\delta\gamma$ and $\varepsilon^{1/m} < \delta\gamma/6Ce$. It follows that (2.3) and (3.1) are in contradiction for large λ, so the proof is complete.

As an application, we obtain an improvement of Theorem 1.4.5 of [3]:

THEOREM 3.5. *If X is an open set in $\mathbb{R}^n$ which is P-convex with respect to singular supports, it follows that the minimum principle is valid for the boundary distance $d(x, \complement X)$ on every affine subspace parallel to a linear subspace with $\sigma_P(V') = 0$.*

We refer to [3] for the terminology and the derivation of Theorem 3.5 from Theorem 3.1 for the case $\Gamma = V$.

4. Fundamental solutions and uniqueness theorems

The first part of Theorem 1.1 can be stated as follows: If $P(D)u \in C^\infty$, $u \in C^\infty(\complement V)$ and $\sigma_P(V') \neq 0$, then $u \in C^\infty$. Thus for general u the singularities of u in $\complement V$ together with all those of $P(D)u$ determine uniquely all the singularities of u. We shall prove more general uniqueness theorems in this section by constructing appropriate fundamental solutions.

We have seen in section 2 that if $\sigma_P(V') \neq 0$ it follows that for some constant $c > 0$ and every $\lambda > 0$

$$\liminf_{\xi\to\infty} \tilde{P}_{V'}(\xi, \lambda \log|\xi|)/\tilde{P}(\xi, \lambda\log|\xi|) > c. \tag{4.1}$$

Our first lemma interprets this condition in terms of the zeros of P. It is interesting to compare it with Lemma 3.4.

LEMMA 4.1. *Let $0 \neq \eta^0 \in V'$, $|\eta^0| = 1$, and let δ, c be fixed positive constants, $\delta < 1$. Then there exist positive constants c_1, γ depending only on δ, c, n, m, such that*

$$\tilde{P}_{V'}(\xi, t)/\tilde{P}(\xi, t) > c \tag{4.2}$$

implies that for some $\theta \in V'$ with $0 < |\theta| < \delta t$ we have

$$|P(\xi + it\eta^0 + z\theta + \zeta)| \geqq c_1\tilde{P}(\xi, t) \text{ if } z \in \mathbb{R},\ |z| = 1;\ \zeta \in \mathbb{C}^n,\ |\zeta| \leqq 2\gamma t. \tag{4.3}$$

PROOF. We may assume that $t = 1$ and can drop the variable t from the notation then. By (4.2) and Taylor's formula we have

$$c\tilde{P}(\xi) \leqq \tilde{P}_{V'}(\xi) \leqq C\tilde{P}_{V'}(\xi + i\eta^0).$$

Lemma 3.1.7 in [1] shows that we can choose $\theta \in V'$ with $0 < |\theta| < \delta$ and

$$\tilde{P}_{V'}(\xi + i\eta^0) \leqq C'|P(\xi + i\eta^0 + z\theta)|, \qquad |z| = 1.$$

For small $|\zeta|$ it follows that when $|z| = 1$

$$|P(\xi + i\eta^0 + z\theta + \zeta)| \geqq c_3\tilde{P}_{V'}(\xi + i\eta^0) - C_1|\zeta|\tilde{P}(\xi + i\eta^0 + z\theta) \geqq (c_4 - C_2|\zeta|)\tilde{P}(\xi).$$

This proves the lemma.

REMARK. Conversely the inequality (4.3) for $\zeta = 0$ already implies an estimate of the form (4.2) so (4.3) expresses all the information given by (4.2).

Before developing further technical details we shall indicate the main ideas in our arguments. To construct a fundamental solution one usually interprets the integral

$$(2\pi)^{-n}\int e^{i\langle x,\zeta\rangle}P(\zeta)^{-1}\,d\zeta$$

by taking it over some cycle, avoiding the zeros of P, which is close to $\mathbb{R}^n$. Sometimes the cycle is taken close to the cycle defined by

$$\xi \to \xi + i\lambda(\log|\xi|)\eta^0$$

where η^0 is fixed in $\mathbb{R}^n \setminus 0$ and λ is large. The modulus of the exponential is then $|\xi|^{-\lambda\langle x,\eta^0\rangle}$ so the fundamental solution becomes roughly $\lambda\langle x,\eta^0\rangle$ times differentiable at x (thus a distribution of order $-\lambda\langle x,\eta^0\rangle$ when $\langle x,\eta^0\rangle<0$). See for example [2, section 5] and the references given there. We shall improve the construction by replacing the value of $e^{i\langle x,\zeta\rangle}/P(\zeta)$ at a point on the cycle by an average over the zero free region provided by Lemma 4.1. This will be done so that Lemma 3.3 can be used to improve the estimates. The number N there will also be taken of the order of magnitude $\log|\xi|$. To achieve this we must work in steps where N is fixed so the integration will be split in countably many pieces by means of a partition of unity similar to those used in [2].

First we shall define for arbitrary (ξ,t) satisfying (4.2) and integers $N>0$ a measure $\mu_{\xi,t}^N$ in $\mathbb{C}^n$ which will replace the Dirac measure at $\xi+it\eta^0$. To do so we use the function Φ_R^N of Lemma 3.3 with n variables and set for $u\in C_0(\mathbb{C}^n)$ with the notations of Lemma 4.1

$$\int u(\zeta)d\mu_{\xi,t}^N(\zeta) = (2\pi)^{-1}\int_0^{2\pi}d\psi\int u(\xi+it\eta^0+e^{i\psi}\theta+\tau)\Phi_{\gamma t}^N(\tau)d\tau.$$

It is clear that θ can be chosen as a measurable function of ξ for every fixed t. By Cauchy's integral formula we have for analytic u

$$\int u(\zeta)d\mu_{\xi,t}^N(\zeta) = \int u(\xi+it\eta^0+\tau)\Phi_{\gamma t}^N(\tau)d\tau. \tag{4.4}$$

Lemma 3.3 gives the estimate

$$\left|\int e^{i\langle x,\zeta\rangle}/P(\zeta)d\mu_{\xi,t}^N(\zeta)\right| \leqq C(3N/\gamma t|x|)^k e^{t(\delta|x'|-\langle x,\eta^0\rangle)}/\tilde{P}(\xi,t), \quad 0\leqq k\leqq N. \tag{4.5}$$

When $t>1$ it follows that

$$(4.6) \qquad \left| \int e^{i\langle x,\zeta\rangle}/P(\zeta)d\mu^N_{\xi,t}(\zeta) \right| \leqq C\exp(t(\delta|x'| - \langle x,\eta^0\rangle) - N) \text{ if } |x| > 3Ne/\gamma t.$$

We shall later on choose N proportional to t and this estimate is then valid in a set independent of t.

Next we shall construct the required partition of unity. First choose as in [2] a function $\phi \in C_0^\infty(\mathbb{C})$ such that $|\mathrm{Re}\,\zeta| < 3/4$ if $\zeta \in \mathrm{supp}\,\phi$, $\Sigma\phi(\zeta - j) = 1$ if $|\mathrm{Im}\,\zeta|$ is sufficiently small and

$$|\partial\phi(\zeta)/\partial\bar{\zeta}| \leqq C_\nu |\mathrm{Im}\,\zeta|^\nu, \ \zeta \in \mathbb{C}, \ \nu = 1,2,\cdots$$

When $\zeta \in \mathbb{C}^n$ and $\langle \mathrm{Im}\,\zeta, \mathrm{Im}\,\zeta\rangle < \langle \mathrm{Re}\,\zeta, \mathrm{Re}\,\zeta\rangle$ we can define $\langle\zeta,\zeta\rangle^{1/4}$ as an analytic function which is positive when $\mathrm{Im}\,\zeta = 0$, and we set

$$\psi_j(\zeta) = \phi(\langle\zeta,\zeta\rangle^{1/4} - j).$$

The following lemma is obvious if we note that $\langle\zeta,\zeta\rangle = |\xi|^2(1 + O(|\eta|/|\xi|))$, with the Euclidean norm. (See also Lemma 4.4 in [2].)

LEMMA 4.2. *There is a positive constant c_2 and a positive integer j_0 such that when $|\mathrm{Im}\,\zeta| < c_2|\mathrm{Re}\,\zeta|^{1/2}$*

(i) $j - 4/5 < |\mathrm{Re}\,\zeta|^{1/2} < j + 4/5$ *if* $\zeta \in \mathrm{supp}\,\psi_j$ *and* $j \geqq j_0$,

(ii) $\sum_{j_0}^{\infty} \psi_j(\zeta) = 1$ *if* $|\zeta|$ *is large,*

(iii) $|\bar{\partial}\psi_j(\zeta)| \leqq C_\nu |\mathrm{Im}\,\zeta|^\nu |\zeta|^{-(\nu+1)/2}$ *if* $j \geqq j_0$, $\quad \nu \geqq 0$.

In particular it follows from (i) and (iii) that $\Sigma|\bar{\partial}\psi_j(\zeta)|$ is rapidly decreasing, if $\zeta \to \infty$ while $|\mathrm{Im}\,\zeta| = O(\log|\zeta|)$ as will be the case in the constructions below. The functions ψ_j will therefore behave essentially as if they were analytic.

Choose two positive numbers λ and ε and set

$$(4.7) \qquad t_j = \lambda \log j^2 = 2\lambda \log j, \qquad N_j = [\varepsilon t_j].$$

By (i) in Lemma 4.2 we have for $j \geqq j_0$

$$(4.8) \qquad |t_j - \lambda\log|\xi|| < C\lambda/j \text{ when } \xi + it_j\eta^0 \in \mathrm{supp}\,\psi_j.$$

With χ_j denoting a minor modification of ψ_j which will be defined later on we set for large j

$$(4.9) \qquad E_j(x) = (2\pi)^{-n} \int \chi_j(\xi + it_j\eta^0)d\xi \int e^{i\langle x,\zeta\rangle}/P(\zeta)d\mu^{N_j}_{\xi,t_j}(\zeta).$$

We shall prove that ΣE_j converges in $\mathscr{D}'$ to a parametrix for P, and we shall study its differentiability properties.

First note that by (4.4) we obtain after differentiation under the integral sign

$$P(D)E_j(x) = (2\pi)^{-n} \int\int \chi_j(\xi + it_j\eta^0)e^{i\langle x,\xi+it_j\eta^0+\tau\rangle}\Phi_{\gamma t_j}^{N_j}(\tau)d\xi d\tau.$$

The integration with respect to ξ can be shifted to the real axis by means of Cauchy's integral formula, which gives $P(D)E_j = v_j + w_j$ where

$$v_j(x) = (2\pi)^{-n} \int\int \chi_j(\xi)e^{i\langle x,\xi+\tau\rangle}\Phi_{\gamma t_j}^{N_j}(\tau)d\xi d\tau,$$

$$w_j(x) = -2i(2\pi)^{-n} \int\int\int_{0<t<t_j} \langle \bar{\partial}\chi_j(\xi + it\eta^0), \eta\rangle e^{i\langle x,\xi+it\eta^0+\tau\rangle}\Phi_{\gamma t_j}^{N_j}(\tau)d\xi dt d\tau.$$

Thus v_j is the inverse Fourier transform of the convolution $\chi_j * \Phi_{\gamma t_j}^{N_j}$. We wish Σv_j to be equal to the Dirac measure δ_0 apart from a C^∞ term. This is true if

(4.10) $\qquad \Sigma\chi_j * \Phi_{\gamma t_j}^{N_j} - 1$ has compact support.

To satisfy this condition we start from the functions ψ_j of Lemma 4.2 and set

$$\chi_j(\xi + i\eta) = \int \psi_j(\xi - \theta + i\eta)\Phi_{\gamma t k}^{N_k}(\theta)d\theta$$

where $k = j+1$ if $|\xi| > j^2$ and $k = j-1$ if $|\xi| < j^2$. Since $\psi_j(\xi + i\eta) = 1$ if $||\xi|^{1/2} - j| < 1/5$ and $|\eta| < c_2|\xi|^{1/2}$, both definitions give $\chi_j(\xi + i\eta) = 1$ if $||\xi|^{1/2} - j| < 1/10$, $|\eta| < c_2|\xi|^{1/2}/2$ and $j \geqq J$ say. It follows that $\chi_j \in C^\infty$ and it is clear that (i)–(iii) in Lemma 4.2 and (4.8) remain valid with ψ_j replaced by χ_j and a change of the constants involved. Since for large $|\xi|$ with $j < |\xi|^{1/2} < j+1$ the sum in (4.10) is equal to $(\psi_j + \psi_{j+1}) * \Phi_{\gamma t_j}^{N_j} * \Phi_{\gamma t_{j+1}}^{N_{j+1}}$, we obtain (4.10) from (i) and (ii) in Lemma 4.2. The fact that (iii) in Lemma 4.2 remains valid with ψ_j replaced by χ_j shows that for all α and ν we have on compact sets

$$D^\alpha w_j = O(j^{-\nu}).$$

Hence $\Sigma_J^\infty w_j \in C^\infty$, and $\Sigma_J^\infty v_j - \delta_0 \in C^\infty$ with convergence in $\mathscr{S}'$ by (4.10).

When $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\langle E_j, u\rangle = (2\pi)^{-n} \int \chi_j(\xi + it_j\eta^0)d\xi \int \hat{u}(-\zeta)/P(\zeta)d\mu_{\xi,t_j}^{N_j}(\zeta),$$

and, since $\hat{u}$ is rapidly decreasing in the support of the integrand, the sum $\Sigma\langle E_j, u\rangle$

is absolutely convergent. Hence $E = \sum_j^\infty E_j$ exists in $\mathscr{D}'$ and $P(D)E - \delta_0 \in C^\infty$. We could therefore subtract a C^∞ function from E and obtain a fundamental solution.

We shall finally use (4.6) and the analogous bounds for the derivatives with respect to x to study the differentiability of E. By (4.8) which is valid with ψ_j replaced by χ_j, we can estimate $|\xi|$ by $\exp(t_j/\lambda)$ and obtain for large j

$$|D^\alpha E_j(x)| \leqq C_\alpha \exp(t_j((n + |\alpha|)/\lambda + \delta|x'| - \langle x,\eta^0\rangle - \varepsilon)) \text{ if } |x| > 3\varepsilon e/\gamma.$$

Hence $\sum D^\alpha E_j$ is uniformly convergent when

$$2(n + |\alpha|) + 2\lambda(\delta|x'| - \langle x,\eta^0\rangle - \varepsilon) < -1, \ |x| > 3\varepsilon e/\gamma.$$

For any ν it follows that $E \in C^\nu$ for large λ in the set defined by

$$\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/20 \leqq 0, \ 3\varepsilon e/\gamma \leqq |x| \leqq 6\varepsilon e/\gamma.$$

Now choose a cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ which is 1 when $|x| < 3\varepsilon e/\gamma$ and 0 when $|x| > 6\varepsilon e/\gamma$, and set $F = \chi E$. Then we have $P(D)F = \delta + \omega$ where $\omega \in C^\infty$ for $|x| < 3\varepsilon e/\gamma$ and $\omega \in C^\nu$ for large λ in the set where $\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/20 \leqq 0$. Replacing $3\varepsilon e/\gamma$ by ε we have proved

THEOREM 4.3. *Assume that V is a linear subspace of $\mathbb{R}^n$ with $\sigma_P(V') \neq 0$. Choose $\eta \in V'$ with $|\eta| = 1$ and δ with $0 < \delta < 1$. For $\varepsilon > 0$ and positive integers ν, one can then find $F_{\varepsilon,\nu} \in \mathscr{E}'(\mathbb{R}^n)$ such that $|x| < 2\varepsilon$ in $\operatorname{supp} F_{\varepsilon,\nu}$, the difference $P(D)F_{\varepsilon,\nu} - \delta_0 \in C^\infty\{x; |x| < \varepsilon\}$ and*

$$F_{\varepsilon,\nu} \in C^\nu\{x; \delta|x'| - \langle x,\eta\rangle - \gamma|x|/20 < 0\}.$$

Here x' is the residue class of x mod V and γ is the constant in Lemma 4.1.

If u is a distribution with $P(D)u = f \in C^\infty$ for $|x| < 3\varepsilon$, we have for $|x| < \varepsilon$

$$u = u * (\delta_0 - P(D)F_{\varepsilon,\nu}) + f * F_{\varepsilon,\nu}.$$

The last term is in C^∞ when $|x| < \varepsilon$. If $u \in C^\infty$ in a neighborhood of

$$\{x; \varepsilon \leqq |x| \leqq 2\varepsilon, \ \delta|x'| + \langle x,\eta\rangle - \gamma|x|/20 \geqq 0\}, \tag{4.11}$$

it follows that $u \in C^\infty$ in a neighborhood of 0 if we let $\nu \to \infty$. This will give the following theorem which contains the first half of Theorem 1.1.

THEOREM 4.4. *Let $\phi_1, \cdots, \phi_\nu \in C^1(X)$ where X is an open set in $\mathbb{R}^n$ and let x^0 be a point in X where $d\phi_1(x^0), \cdots, d\phi_\nu(x^0)$ are linearly independent. Assume that $\sigma_P(W) \neq 0$ for the space W spanned by $d\phi_1(x^0), \cdots, d\phi_\nu(x^0)$. If $u \in \mathscr{D}'(X)$, $P(D)u \in C^\infty(X)$ and $u \in C^\infty(X_-)$,*

$$X_- = \{x \in X;\ \phi_j(x) < \phi_j(x^0) \ \textit{for some}\ j = 1,\cdots,\nu\},$$

then $u \in C^\infty$ *in a neighborhood of* x^0.

PROOF. We may assume that $x^0 = 0$ and that $\phi_j(x) = x_j + o(|x|)$, as $x \to 0$. Take $\eta = (-1,\cdots,-1,0,\cdots,0)$ with ν coordinates equal to -1. When $x \in \operatorname{sing\,supp} u$ we have by hypothesis $x_j \geqq -o(|x|)$ for $j = 1,\cdots,\nu$, hence $|x_j - |x_j|| = o(|x|)$ then, which gives

$$\begin{aligned}\delta|x'| + \langle x,\eta\rangle - \gamma|x|/20 &\leqq \delta \sum_1^\nu |x_j| - \sum_1^\nu |x_j| + (o(|x|) - \gamma|x|/20)\\ &\leqq o(|x|) - \gamma|x|/20 < 0\end{aligned}$$

if $\varepsilon < |x| < 2\varepsilon$ and ε is small. It follows that $u \in C^\infty$ in the set (4.11) and the proof is therefore complete.

We can now prove another result stated in the introduction:

THEOREM 4.5. *Let* $X_1 \subset X_2$ *be open convex sets in* $\mathbb{R}^n$. *Then an open set* $X \subset X_2$ *has the property*

$$u \in \mathscr{D}'(X_2),\ Pu \in C^\infty(X_2),\ u \in C^\infty(X_1) \Rightarrow u \in C^\infty(X) \tag{4.12}$$

if and only if for every hyperplane H *with* $\sigma_P(H') = 0$ *the set* X_1 *intersects every affine hyperplane parallel to* H *which meets* X.

PROOF. The necessity follows immediately from Theorem 1.1. The sufficiency is proved by substituting Theorem 4.4 with $\nu = 1$ for Holmgren's uniqueness theorem in the proof of Theorem 5.3.3 in [1].

Theorem 4.3 also gives a converse of Theorem 3.1:

THEOREM 4.6. *Let* Γ *be a closed convex set in* $\mathbb{R}^n$ *and let* V *be the largest vector space with* $\Gamma + V = \Gamma$. *Then* $u \in \mathscr{D}'(\mathbb{R}^n)$, sing supp $u \subset \Gamma$ *implies* $u \in C^\infty(\mathbb{R}^n)$ *if* $\sigma_P(V') \neq 0$.

PROOF. If $0 \notin \Gamma$ there is a proper cone in $\mathbb{R}^n/V$ containing the image of Γ there. We may therefore assume that Γ is the inverse image of such a cone. The set of points where Theorem 4.3 can be used to show that $u \in C^\infty$ is obviously a cone since ε may be any positive number. In the proof of Theorem 4.4 we saw that it contains a neighborhood of 0 so it is all of $\mathbb{R}^n$.

Minimal linear spaces V with $\sigma_P(V') = 0$ are minimal carriers of singularities also among sets which are not linear:

THEOREM 4.7. *Let* V *be a linear subspace of* $\mathbb{R}^n$ *such that* $\sigma_P(V') = 0$ *but*

$\sigma_P(W') \neq 0$ *for every linear subspace* $W \subsetneq V$. *If* $P(D)u \in C^\infty$ *and* sing supp $u \subset V$, *it follows that either* sing supp $u = V$ *or* $u \in C^\infty$.

PROOF. Let V be defined by $x' = 0$ and assume that for some $r > 0$, we have $u \in C^\infty$ when $|x| < r$. Thus $u \in C^\infty$ at (x', x'') if $|x''| < r$ or $|x'| \neq 0$. If a is a linear function of x' it follows that $u \in C^\infty$ where

$$\phi(x) = |x''|^2 - r^2 - a(x') < 0.$$

(Here we are using the Euclidean norm.) In fact, if $\phi(x) < 0$ we have either $x' \neq 0$ or $|x''| < r$. We have grad $\phi = (-a, 2x'')$ so if $x_0 = (0, x_0'')$, $|x_0''| = r$, the plane spanned by these gradiends is $V' \oplus \mathbb{R}x_0$. The orthogonal space $W = \{x \in V; \langle x, x_0 \rangle = 0\}$ and since $\sigma_P(W') \neq 0$ it follows from Theorem 4.3 that $u \in C^\infty$ in a neighborhood of the closed ball of radius r. Hence $u \in C^\infty$.

Theorems 4.6 and 4.7 may seem to indicate that for solutions of $P(D)u = 0$, the singularities must propagate along linear spaces in the sense that for every $x \in$ sing supp u there is a linear space V with $\sigma_P(V') = 0$ such that $\{x\} + V \subset$ sing supp u. This is known to be true in a number of cases (see [2], [3]) and others will be given below, but it is false in general. For, consider the differential operator $P(D) = D_2 D_3$ in $\mathbb{R}^3$. Set $u = \delta(x_1)(f(x_2) + g(x_3))$ where $f(x_2) = 1$ when $|x_2| < 1$ and 0 otherwise, $g(x_3) = -1$ when $2 < x_3 < 3$ and 0 otherwise. Then we have $P(D)u = 0$ and $0 \in$ sing supp u but no straight line through 0 is contained in sing supp u.

A positive result to be improved in section 5 is given in

THEOREM 4.8. *Let* V *be the intersection of all linear spaces* W *with* $\sigma_P(W') = 0$. *If* $u \in \mathscr{D}'(X)$, $P(D)u \in C^\infty(X)$ *and* $x \in$ sing supp u, *it follows that the component of* x *in* $(\{x\} + V) \cap X$ *also belongs to* sing supp u.

PROOF. If there exists a polygon with vertices $x = x_0, x_1, \cdots, x_N$ in $(\{x\} + V) \cap X$ such that $x_N \notin$ sing supp u, then repeated application of Theorem 4.5 shows that $x_{N-1}, \cdots, x_0 \notin$ sing supp u. This proves the theorem.

If $V = \{0\}$, the theorem is of course trivial. On the other hand, if $\sigma_P(V') = 0$, we obtain complete information on the possible singular support of a solution.

5. Geometric mean estimates

In this section we shall consider the quotient $\tilde{P}_V(\xi, t)/\tilde{P}(\xi, t)$ for all ξ so we introduce

$$\sigma_P^0(V) = \inf_{t > 1} \inf_{\xi} \tilde{P}_V(\xi, t)/\tilde{P}(\xi, t). \tag{5.1}$$

THEOREM 5.1. *Let X be an open subset of $\mathbb{R}^n$ with $0 \in X$ and V a linear subspace with orthogonal space V'. Then the following conditions are equivalent:*

(i) *For every semi-norm q in $C^\infty(X)$ with support sufficiently near 0, there are semi-norms q_1 in $C(X)$ and q_2 in $C^\infty(X \setminus V)$ and a constant ρ with $0 < \rho < 1$ such that*

$$q(u) \leqq (q_1(u) + q_2(u))^{1-\rho} q_2(u)^\rho; \ u \in C^\infty(X), \qquad P(D)u = 0.$$

(ii) *There exist semi-norms q_1 and q_2 in $C^\infty(X)$ and $C^\infty(X \setminus V)$ respectively and a constant ρ with $0 < \rho < 1$ such that*

$$|u(0)| \leqq q_1(u)^{1-\rho} q_2(u)^\rho; \ u \in C^\infty(X), \qquad P(D)u = 0.$$

(iii) $\sigma_P^0(V') \neq 0$.

(iv) *If Y is an open set in $\mathbb{R}^k$ for some k, then*

$$u \in \mathscr{D}'(X \times Y),\ P(D)u = 0,\ u \in C^\infty((X \setminus V) \times Y) \Rightarrow u \in C^\infty(X \times Y). \tag{5.2}$$

(v) (5.2) *is valid for some open non-void set $Y \subset \mathbb{R}^k$, $\quad k \geqq 1$.*

PROOF. We have trivial implications (i) $\Rightarrow$ (ii) (and (iv) $\Rightarrow$ (v)). Furthermore Theorem 1.1 shows that (iii), (iv), (v) are equivalent. By inspecting the arguments already used in sections 3 and 4 we shall show that (ii) $\Rightarrow$ (iii) and that (iii) $\Rightarrow$ (i). (The implication (v) $\Rightarrow$ (i) is also easily obtained by functional analysis so we would not have to use Theorem 1.1.)

(ii) $\Rightarrow$ (iii). Assume that $\sigma_P^0(V') = 0$. If

$$a(t) = \inf_\xi \tilde{P}_{V'}(\xi, t)/\tilde{P}(\xi, t),$$

we have $a(st) \leqq Cs^m a(t)$, $s \geqq 1$, so the hypothesis $\inf a(t) = 0$ implies $\liminf_{t\to\infty} a(t) = 0$. Hence it follows from the Tarski-Seidenberg theorem that $a(t) \to 0$ as $t \to \infty$ and that for every $\varepsilon > 0$ one can find r_1, t_1, κ so that when $t > t_1$ there is some ξ with $|\xi| < r_1 t^\kappa$ for which

$$\tilde{P}_{V'}(\xi, t)/\tilde{P}(\xi, t) < \varepsilon. \tag{5.3}$$

For such ξ and t we consider the solution (3.5) of the equation $P(D)u = 0$. We have $u(0) = 1$ and for some μ

$$q_1(u) \leqq C(t + |\xi|)^\mu \exp(Ct\varepsilon^{1/m}).$$

The semi-norm $q_2(u)$ is a bound for ν derivatives of u in a compact set $K_2 \subset \complement V$, so $|x'| > \delta > 0$ in K_2 if $V = \{(x', x''), x' = 0\}$. Hence we can use the estimate (3.6) and obtain with the same choice of N as in section 3

$$q_2(u) \leqq C(t + |\xi|)^\nu \exp(Ct\varepsilon^{1/m} - \delta\gamma t/3e).$$

When ε is so small that

$$(1 - \rho)C\varepsilon^{1/m} + \rho(C\varepsilon^{1/m} - \delta\gamma/3e) < 0,$$

that is, $C\varepsilon^{1/m} < \rho\delta\gamma/3e$, we conclude that (ii) is not valid.

To prove the remaining implication (iii) $\Rightarrow$ (i), we shall reconsider the parametrix construction in section 4. It is considerably simpler now since Lemma 4.1 and the definition of $\mu_{\xi,t}^N$ are valid for all ξ and $t > 1$. With $t > 1$ and a positive integer N we set

$$E_{t,N}(x) = (2\pi)^{-n} \int d\xi \int e^{i\langle x,\zeta\rangle}/P(\zeta)d\mu_{\xi,t}^N(\zeta). \tag{5.4}$$

The integral converges in $\mathscr{D}'$, that is, $E_{t,N}$ is actually defined by

$$\langle E_{t,N}, u\rangle = (2\pi)^{-n}\int d\xi \int \hat{u}(-\zeta)/P(\zeta)d\mu_{\xi,t}^N(\zeta),\ u \in C_0^\infty(\mathbb{R}^n). \tag{5.5}$$

$E_{t,N}$ is a fundamental solution, for by (4.4) and Cauchy's integral formula

$$\begin{aligned}\langle E_{t,N}, P(-D)u\rangle &= (2\pi)^{-n}\int d\xi \int \hat{u}(-\zeta)d\mu_{\xi,t}^N(\zeta)\\ &= (2\pi)^{-n}\int\int \hat{u}(-\xi - it\eta^0 - \tau)\Phi_{\gamma t}^N(\tau)d\xi d\tau\\ &= (2\pi)^{-n}\int \hat{u}(\xi)d\xi \int \Phi_{\gamma t}^N(\tau)d\tau = u(0).\end{aligned}$$

The proof of Theorem 3.1.2 in [1] shows that $E_{t,N}$ is regular in the sense of [2].

When estimating $E_{t,N}$ it is convenient to assume that

$$\int \tilde{P}(\xi)^{-1}\, d\xi < \infty.$$

This condition is no restriction in the proof of Theorem 5.1 since it is always fulfilled after multiplication of P by an elliptic factor of degree $> n$. (See section 6.) By (4.5) the integral (5.4) is then convergent and

$$|E_{t,N}(x)| \leqq C\exp(t(\delta|x'| - \langle x,\eta^0\rangle) - N) \text{ if } |x| > 3Ne/\gamma t.$$

Let ε be a positive number and set $N = [\gamma\varepsilon t/3e]$. Then we obtain for large t

$$|E_{t,N}(x)| \leqq C\exp t(\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/18) \text{ if } \varepsilon < |x| < 2\varepsilon.$$

Let $\chi \in C_0^\infty$ be equal to 1 when $|x| < 4\varepsilon/3$ and 0 when $|x| > 5\varepsilon/3$. Then choose

ψ_1 and $\psi_2 \in C_0^\infty$ with support in $\{x; \varepsilon < |x| < 2\varepsilon\}$ so that $\psi_1 + \psi_2 = 1$ in a neighborhood of supp $d\chi$ and

$$\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/18 < 0 \text{ in supp } \psi_1$$
$$\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/20 > 0 \text{ in supp}\, \psi_2 .$$

Set $\omega_j = -\psi_j P(D)(\chi E)$. Then it is clear that $P(D)(\chi E) = \delta_0 - \omega_1 - \omega_2$, and for some positive constants a and A we have as $t \to \infty$

$$e^{at}\omega_1 \text{ and } e^{-At}\omega_2 \text{ are bounded in } \mathscr{D}'^m, \tag{5.6}$$

$$\delta|x'| - \langle x,\eta^0\rangle - \gamma|x|/20 > 0 \text{ in supp}\, \omega_2 . \tag{5.7}$$

If $P(D)u = 0$ in X and ε is small, we obtain $u = \omega_1 * u + \omega_2 * u$ in a neighborhood of 0. For any semi-norm q in C^∞ of a small neighborhod of 0 it follows that

$$q(u) \leqq e^{-at}q_1(u) + e^{At}q_2(u)$$

where q_1 is a semi-norm in $C^\infty(X)$ and q_2 is a semi-norm in $C^\infty(N)$,

$$N = \{x; \varepsilon < |x| < 2\varepsilon, \delta|x'| + \langle x,\eta^0\rangle - \gamma|x|/20 > 0\}.$$

We may assume that $q_1 \geqq q_2$. Choosing t so that $e^{(A+a)t} = q_1(u)/q_2(u)$ we obtain with $\rho = a/(A+a)$

$$q(u) \leqq 2q_1(u)^{1-\rho}q_2(u)^\rho. \tag{5.8}$$

It is now easy to prove a result containing the implication (iii) $\Rightarrow$ (i) in Theorem 5.1 :

THEOREM 5.2. *Let $\phi_1, \cdots, \phi_\nu \in C^1(X)$ where X is an open set in $\mathbb{R}^n$, and let x^0 be a point in X where $d\phi_1(x^0), \cdots, d\phi_\nu(x^0)$ are linearly independent. Assume that $\sigma_P^0(W) \neq 0$ for the plane W spanned by $d\phi_1(x^0), \cdots, d\phi_\nu(x^0)$. If U is a sufficiently small neighborhood of 0 and q a semi-norm in $C^\infty(U)$, one can find semi-norms q_1 in $C(X)$ and q_2 in $C^\infty(X_-)$,*

$$X_- = \{x \in X;\ \phi_j(x) < \phi_j(x^0) \text{ for some } j\},$$

and a number ρ with $0 < \rho < 1$ such that

$$q(u) \leqq (q_1(u) + q_2(u))^{1-\rho}q_2(u)^\rho;\ u \in C^\infty(X),\ P(D)u = 0. \tag{5.9}$$

PROOF. As in the proof of Theorem 4.4 it follows from (5.8) that

$$q(u) \leqq q_0(u)^{1-\rho} q_2(u)^\rho$$

where q_0 is a semi-norm in $C^\infty(X)$. Replacing X by a smaller neighborhood Y of 0 we can assume that q_0 is a semi-norm in Y. If Y is sufficiently small we have

$$q_0(u) \leqq q_1(u) + q_3(u),\ u \in C^\infty(X),\ P(D)u = 0,$$

where q_1 is a semi-norm in $C(X)$ and q_3 a semi-norm of the same kind as q_2. In fact, by Theorem 4.4 we can choose Y so that $P(D)u = 0$ in X and $u \in C^\infty(X_-)$ implies $u \in C^\infty(Y)$, and then the assertion follows from the closed graph theorem. This completes the proof.

If we combine Theorems 5.1 and 5.2 with the proof of Theorem 4.5, we obtain

THEOREM 5.3. *Let $X_1 \subset X_2$ be open convex sets in $\mathbb{R}^n$ and let X be an open subset of X_2. Then the following conditions are equivalent:*

(i) *For every hyperplane H with $\sigma_P^0(H) = 0$ the set X_1 intersects every affine hyperplane parallel to H which meets X.*

(ii) *For every semi-norm q in $C^\infty(X)$ there exist semi-norms q_1 in $C^\infty(X_1)$ and q_2 in $C(X_2)$ such that for some $\rho > 0$*

$$q(u) \leqq (q_1(u) + q_2(u))^{1-\rho} q_1(u)^\rho;\ u \in C^\infty(X_2),\ P(D)u = 0. \tag{5.10}$$

Similarly repetition of the proof of Theorem 4.8 gives

THEOREM 5.4. *Let V be a subspace of $\mathbb{R}^n$ such that $\sigma_P^0(W) \neq 0$ when W is any subspace which is not orthogonal to V. Let X_2 be an open set in $\mathbb{R}^n$, $x_0 \in X_2$, and let X_1 be an open subset which meets the component of x_0 in $X_2 \cap (V + \{x_0\})$. Then there is a neighborhood X of x_0 such that* (ii) *in Theorem* 5.3 *is valid.*

Inspection of the proof shows that the semi-norms in (5.10) will only depend on m and n in addition to a lower bound for $\sigma_P^0(W)$ when $d(W, V')$ is bounded away from 0. (See the introduction for this notation.) If P is replaced by a power of P one can keep the constant γ in Lemma 4.1 and therefore the exponent ρ in (5.10), and it suffices to multiply the semi-norms in (5.10) by a constant factor. This makes it possible to repeat the arguments used in section 1.5 of [3] to extend Theorem 1.5.1 there. Introduce

$$|W, V| = \sup_{x \in W,\, \xi \in V} |\langle x, \xi\rangle| / |x|\,|\xi|$$

which is equivalent to $d(W, V')$. Let B be a compact family of linear subspaces of $\mathbb{R}^n$ and denote by $L(P)$ the set of all localizations of P at infinity as defined in [2]. Thus the elements of $L(P)$ are the limits of the polynomials $\xi \to P(\xi + \eta)/\tilde{P}(\eta)$ as

$\eta \to \infty$. Now assume that for every $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that for every $Q \in L(P)$ there is some $V \in B$ for which

$$\sigma_Q^0(W) \geqq c(\varepsilon) \text{ when } |W, V| \geqq \varepsilon. \tag{5.11}$$

Under these hypotheses we have

THEOREM 5.5. *If $u \in \mathscr{D}'(X)$ and $Pu \in C^\infty(X)$ it follows that for every $x \in \operatorname{sing\,supp} u$ there is some $V \in B$ such that the component of x in $X \cap (V + \{x\})$ also belongs to* $\operatorname{sing\,supp} u$.

We omit the rather tedious details of the proof. The theorem is of course trivial if $\{0\} \in B$ so the result is void in general. However, it contains Theorem 4.8 above as well as Theorem 1.5.1 of [3] and Theorem 7.2 of [2].

6. Remarks and examples

If $L(P)$ is the set of all localizations of P at infinity, it is clear that

$$\sigma_P^0(V) \leqq \sigma_P(V) = \inf_{Q \in L(P)} \sigma_Q^0(V). \tag{6.1}$$

We shall therefore begin by studying only the function $\sigma_P^0(V)$.

LEMMA 6.1. *If p is the principal part of P, then $\sigma_p^0 \geqq \sigma_P^0$.*

PROOF. By definition we have

$$\tilde{P}_V(\xi, t) \geqq \sigma_P^0(V)\tilde{P}(\xi, t), \qquad t \geqq 1.$$

If we replace ξ, t by $s\xi$, st and let $s \to +\infty$ after division by s^m, it follows that

$$\tilde{p}_V(\xi, t) \geqq \sigma_P^0(V)\tilde{p}(\xi, t)$$

which proves the lemma.

LEMMA 6.2. *If p is a homogeneous polynomial then $\sigma_p^0(V) > 0$ if and only if neither p nor any localization of p at infinity vanishes identically in V.*

PROOF. The necessity is obvious since $\sigma_p^0(V) > 0$ implies $\sigma_q^0(V) > 0$ for every localization q in view of (6.1). To prove the sufficiency it is enough to show that the inequality

$$\tilde{p}(\xi, t) \leqq C\tilde{p}_V(\xi, t) \tag{6.2}$$

is valid when $t = 1$, for both sides are homogeneous in (ξ, t). When $t \neq 0$ we have $\tilde{p}_V(\xi, t) > 0$ because $p(\xi + \eta)$ would otherwise vanish identically for $\eta \in V$

so the highest order term $p(\eta)$ with respect to η would have to vanish in V. Hence (6.2) is valid when $t = 1$ and ξ is in a compact set. If (6.2) were not valid we could therefore find a sequence $\xi_j \to \infty$ such that $\tilde{p}_V(\xi_j, 1)/\tilde{p}(\xi_j, 1) \to 0$. If q is a corresponding localization at infinity we have $q(\eta) = 0$ for all $\eta \in V$, which contradicts the hypothesis.

REMARK. If $p = 0$ in V then the localization of p at infinity in V also vanishes in V so the condition that $p \not\equiv 0$ in V could be dropped.

Combination of (6.1) with Lemmas 6.1 and 6.2 gives

THEOREM 6.3. *If $\sigma_P(V) \neq 0$ it follows that the principal part of Q does not vanish identically in W if $Q \in L(P)$ and W is a linear space with* $\dim W = \dim V$ *sufficiently close to V. If $\sigma_P^0(V) \neq 0$ the principal symbol of P does not vanish identically in W either.*

REMARK. When $n = 2$ it is easy to show that conversely $\sigma_P^0(V) \neq 0$ if the principal symbol of P is not identically 0 in V.

If $Q \in L(P)$ is not a constant and if

$$\Lambda(Q) = \{\eta \in \mathbb{R}^n;\ Q(\xi + t\eta) \equiv Q(\xi)\},$$

which is a linear space, it follows that $\sigma_P(\Lambda(Q)) = 0$. The same is true for the space generated by $\Lambda(Q)$ and a real zero of the principal part of Q. By Theorem 1.1 it follows that the equation $P(D)u = 0$ has a solution with $\operatorname{sing\,supp} u = \Lambda'(Q)$ or a characteristic hyperplane in $\Lambda'(Q)$ (with respect to Q). The singular support may also be taken as the limit of such spaces. Thus Theorem 1.1 improves the results proved in section 3 of [2]. In section 6 of [2] we concluded that unique continuation of the singularities of solutions of $P(D)u = 0$ across the hyperplane $\langle x, N\rangle = 0$ requires that a neighborhood of N is non-characteristic for every $Q \in L(P)$. This is also a special case of Theorem 1.1 in view of Theorem 6.3. We shall now give examples which show that the condition is not always sufficient.

EXAMPLE 6.4. Let q be a real homogeneous polynomial and V a linear subspace of $\mathbb{R}^n$, $n \geqq 3$, such that q does not vanish identically in V but $\sigma_q^0(V) = 0$. A sufficient condition for this is that $0 \neq q'(\eta) \in V'$ for some real $\eta \neq 0$ with $q(\eta) = 0$. In fact, $\xi \to \langle \xi, q'(\eta)\rangle$ is then a localization of q at infinity. If $\xi_n = 0$ in V and $n \geqq 3$ we may therefore take $q(\xi) = \xi_1^2 + \cdots + \xi_{n-2}^2 - \xi_{n-1}\xi_n$, $\eta = (0, \cdots, 0, 1, 0)$.

By Theorem 4.1.9 in [1] we can choose a hypoelliptic operator $P(D)$ such that the principal part is q^4. Since $\sigma_q^0(V) = 0$ it follows from Lemma 6.1 that $\sigma_P^0(V) = 0$, although the principal part of P does not vanish identically in V, if V is not generated by η. In particular we conclude that in general the continuation of solutions of a hypoelliptic equation across a non-characteristic hyperplane is not "Hölder continuous" if the number of variables exceeds 2.

EXAMPLE 6.5. Let $P(\xi)$, $\xi \in \mathbb{R}^n$, be independent of ξ_n but as a function of $\xi_1, \cdots, \xi_{n-1}$ be the hypoelliptic polynomial in example 6.4. The only localizations of P at infinity are then constants or translations of P, so their principal symbols do not vanish identically in W if W is close to V. (We include the ξ_n axis in V also.) But $\sigma_P(V) = \sigma_P^0(V) = 0$, which shows that to examine if $\sigma_P(V) \neq 0$ it does not suffice to consider the characteristics of all localizations at infinity.

The results proved in this paper agree well with the classification of differential operators defined in [1, section 3.3].

THEOREM 6.6. *If P' and P'' are equally strong then $\sigma_{P'}(V) = 0$ is equivalent to $\sigma_{P''}(V) = 0$.*

PROOF. By Theorem 3.3.2 in [1] the hypothesis implies that

$$C_1 \leqq \tilde{P}''(\xi,t)/\tilde{P}'(\xi,t) \leqq C_2; \ \xi \in \mathbb{R}^n, \ t > 1.$$

From section 2 we know that $\sigma_{P'}(V) = 0$ if and only if for some sequences $\xi_j \to \infty$ in $\mathbb{R}^n$ and $t_j \to \infty$ in $\mathbb{R}$ with $t_j = O(|\xi_j|^\varepsilon)$ for every $\varepsilon > 0$

$$\tilde{P}'_V(\xi_j, t_j)/\tilde{P}'(\xi_j, t_j) \to 0.$$

Passing to a subsequence we may assume that the limits

$$Q'(\xi) = \lim_{j \to \infty} P'(\xi_j + t_j\xi)/\tilde{P}'(\xi_j, t_j), Q''(\xi) = \lim_{j \to \infty} P''(\xi_j + t_j\xi)/\tilde{P}'(\xi_j, t_j)$$

exist. In fact, the supremum of $|P'(\xi_j + t_j\xi)|/\tilde{P}'(\xi_j, t_j)$ when ξ varies over the unit ball is 1, and that of $|P''(\xi_j + t_j\xi)|/\tilde{P}'(\xi_j, t_j)$ lies between C_1 and C_2. The limits are therefore not identically 0 and

$$C_1\tilde{Q}'(\xi,t) \leqq \tilde{Q}''(\xi,t) \leqq C_2\tilde{Q}'(\xi,t), \qquad t > 0.$$

Letting $t \to 0$ we conclude that $Q'' = 0$ when $Q' = 0$. Since $Q' = 0$ in V it follows hat $Q'' = 0$ in V, so that $\tilde{P}''_V(\xi_j, t_j)/\tilde{P}'(\xi_j, t_j) \to 0$ as $j \to \infty$. Hence $\sigma_{P''}(V) = 0$ and the theorem is proved.

THEOREM 6.7. *If $P = P_1P_2$ then $\sigma_P(V) > 0$ (resp. $\sigma_P^0(V) > 0$) if and only if $\sigma_{P_1}(V)\sigma_{P_2}(V) > 0$ (resp. $\sigma_{P_1}^0(V)\sigma_{P_2}^0(V) > 0$).*

PROOF. This is an immediate consequence of Lemma 3.3.1 in [1].

THEOREM 6.8. *If P is semi-elliptic then* $\sigma_P^0(V) > 0$ *if and only if the principal part of P does not vanish identically in V.*

PROOF. The necessity follows from Theorem 6.3. To prove the sufficiency we recall that semi-ellipticity means that for certain positive integers $m_1, \cdots, m_n$ we have with $|\alpha : m| = \Sigma \alpha_j / m_j$

$$P(\xi) = \sum_{|\alpha:m| \leqq 1} a_\alpha \xi^\alpha$$

where

$$p(\xi) = \sum_{|\alpha:m| = 1} a_\alpha \xi^\alpha \neq 0, \; 0 \neq \xi \in \mathbb{R}^n.$$

Assume that $m_1 = \cdots = m_\nu > m_{\nu+1} \geqq \cdots \geqq m_n$, and set $x' = (x_1, \cdots, x_\nu)$, $x'' = (x_{\nu+1}, \cdots, x_n)$. We wish to prove that if $N = (N', N''), N' \neq 0$, then

$$\sum_\alpha |P^{(\alpha)}(\xi)|^2 t^{2|\alpha|} \leqq C \sum_j |\langle D, N \rangle^j P(\xi)|^2 t^{2j}, \quad t \geqq 1. \tag{6.3}$$

To do so we note that if we assign the weight $|\alpha : m| + j/m_1$ to $\xi^\alpha t^j$, then all terms are of weight $\leqq 2$ and the terms of weight 2 are

$$\Sigma |\langle D, N' \rangle^j p(\xi)|^2 t^{2j}$$

in the right hand side. Since $N' \neq 0$ this sum is $\neq 0$ when $(\xi, t) \neq 0$. Hence

$$\sum_1^n |\xi_j|^{2m_j} + t^{2m_1} \leqq C \Sigma |\langle D, N' \rangle^j p(\xi)|^2 t^{2j}.$$

It follows that (6.3) is valid for another C if $t + |\xi|$ is large enough, which proves the theorem.

That "Hölder estimates" are valid for the continuation of solutions of semi-elliptic equations across a non-characteristic surface also follows from the arguments of F. John [6] in view of the analyticity of the solutions along the intersection of the characteristic planes proved in [1, section 4.4].

Next we consider operators with simple characteristics although the results of [2], [3] are already complete for them.

THEOREM 6.9. *If P is of principal type, we have* $\sigma_P(V) > 0$ *if and only if* $\sigma_P^0(V) > 0$ *and this is true if and only if*

$$p'(\eta) \notin V' \text{ when } \eta \in \mathbb{R}^n \setminus 0 \text{ and } p(\eta) = 0.$$

PROOF. Since an operator of principal type is as strong as its principal part

([1, Theorem 3.3.7]) we may assume that P is homogeneous in view of Theorem 6.6. But then the theorem follows from Lemma 6.2 and the remark following the lemma.

Finally we give an example containing some new information.

EXAMPLE 6.10. Let p be a real homogeneous polynomial of degree m with $p'(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus 0$, and let

$$P = p^2 + q + r$$

where q is homogeneous of degree $2m-1$ and r is of degree $2m-2$. Then

a) $\sigma_P^0(V) > 0$ if and only if $p'(\xi) \notin V'$ for all $\xi \in \mathbb{R}^n \setminus 0$ with $p(\xi) = 0$.

b) $\sigma_P(V) > 0$ if and only if $p'(\xi) \notin V'$ for all $\xi \in \mathbb{R}^n \setminus 0$ with $p(\xi) = \operatorname{Im} q(\xi) = 0$.

Since $\sigma_P^0(V) > 0 \Rightarrow \sigma_p^0(V) > 0$ the necessity follows from Theorem 6.9 in case a). To prove the necessity in case b) we assume that $\eta \in \mathbb{R}^n \setminus 0$, $p(\eta) = \operatorname{Im} q(\eta) = 0$. Replacing η by $-\eta$ if necessary we may assume that $q(\eta) \leqq 0$. If $q(\eta) = 0$ we note that

$$P(\xi + t\eta)t^{2-2m} \to \langle p'(\eta), \xi\rangle^2 + \langle q'(\eta), \xi\rangle + c, \qquad t \to \infty,$$

so the right hand side is a localization of P at infinity and $p'(\eta) \notin V'$ by Theorem 6.3. If $q(\eta) < 0$, on the other hand, we choose $\theta \in \mathbb{R}^n \setminus 0$ so that $\langle p'(\eta), \theta\rangle^2 + q(\eta) = 0$ and obtain

$$P(\xi + t^2\eta + t\theta)t^{3-4m} \to 2\langle p'(\eta), \xi\rangle\langle p'(\eta), \theta\rangle + c$$

and conclude again that $p'(\eta) \notin V'$.

In proving the sufficiency, we assume that V is defined by $\xi'' = (\xi_{\nu+1}, \cdots, \xi_n) = 0$. In case a) we note that p does not vanish identically in V since $p'(\xi)$ would then be in V' for all $\xi \in V$. Hence $\tilde{P}_V(\xi, t) \geqq ct^{2m}$ so there is nothing to prove except when $t / |\xi|$ is small. Then we have to prove that for $t > 1$

$$|P(\xi)| + t|p(\xi)||\xi|^{m-1} + t^2|\xi|^{2m-2} \leqq C\tilde{P}_V(\xi, t). \tag{6.4}$$

The estimate of $|P(\xi)|$ is obvious and it implies the others outside a conic neighborhood of the zeros of p. Since

$$\sum_1^{\nu} \partial^2 p^2/\partial\xi_j^2 = 2\sum_1^{\nu} |\partial p/\partial\xi_j|^2 > 0$$

at the zeros of p, we obtain the required estimate of $t^2|\xi|^{2m-2}$ in a conic neighborhood Γ where we also require that $\partial p/\partial\xi' \neq 0$. Hence

$$t\sum_1^\nu |p\partial p/\partial\xi_j| \leqq t\sum_1^\nu |2p\partial p/\partial\xi_j + \partial q/\partial\xi_j| + Ct|\xi|^{2m-2} \leqq C'\tilde{P}_V(\xi, t)$$

which gives (6.4) in Γ. In case b) we have to supplement the preceding argument by observing that if Γ' is a closed cone where $\operatorname{Im} q \neq 0$, we have $|\xi|^{2m-1} + |p(\xi)|^2 \leqq C|P(\xi)|$ in Γ' when ξ is large. Hence $t|p(\xi)||\xi|^{m-1} \leqq C|P(\xi)|$ when $t^2 \leqq |\xi|$ and $\xi \in \Gamma'$.

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Lund University
Sweden